Nithyanandan Thyagarajan (CSIRO)
26-Oct-2021

Coherence in radio interferometry

1. What is coherence?
2. Where do we make use of coherence in radio interferometry?
3. What are the assumptions and implications?
4. Potential applications and requirements.

1. What is coherence?

Consider a field in some coordinate, space and/or time, electric fields, $E(\vec{r},t)$, for example.

We define coherence generally as

$$J(\vec{r}_1,\vec{r}_2,t_1,t_2) = \langle E(\vec{r}_1,t_1)\, E^*(\vec{r}_2,t_2)\rangle \qquad (1)$$

and is called as <u>mutual coherence</u>.

The fields are said to be incoherent when

$$\langle E(\vec{r}_1,t_1)\, E^*(\vec{r}_2,t_2)\rangle = 0$$

Nithyanandan.Thyagarajan@csiro.au

2. Where do we use coherence in radio interferometry?

Visibilities!

$$V_{ab} = \langle E(\vec{r}_a) E^*(\vec{r}_b) \rangle \quad (2)$$

The well-known Van Cittert-Zernike (VCZ) theorem relates the visibility (spatial coherence of the E-fields measured in the aperture) to the intensities on the sky.

$$V_{ab} = \int I(\hat{s})\, e^{-i\frac{2\pi}{\lambda}\hat{s}\cdot(\vec{r}_a - \vec{r}_b)}\, d\Omega \quad (3)$$

where,

$$I(\hat{s}) = \begin{cases} \langle \mathcal{E}(\hat{s}_1)\, \mathcal{E}^*(\hat{s}_2) \rangle, & \hat{s}_1 = \hat{s}_2 \\ 0, & \hat{s}_1 \neq \hat{s}_2 \end{cases} \quad (4)$$

$$\Rightarrow I(\hat{s}) = \langle |\mathcal{E}(\hat{s})|^2 \rangle$$

That is, the E-fields on the image plane are incoherent. This is an assumption that was made in order to derive the familiar form of the visibility in the radio interferometry measurement equation.

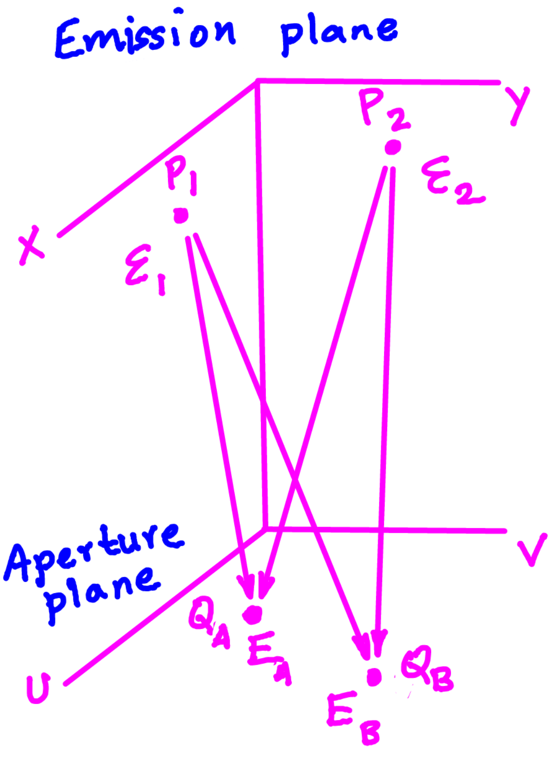
More on this later!

Nithyanandan Thyagarajan (CSIRO)
26-Dec-2021

Monochromatic radiation and coherence

- Why do we hear of quasi-monochromatic treatment of EM radiation in interferometry and not purely monochromatic radiation?
- And what is its relation to coherence?

Consider monochromatic radiation from two sources on the sky.



At P_1, $\mathcal{E}_1 = \xi_1 e^{i(\omega_1 t - \theta(t))}$

At P_2, $\mathcal{E}_2 = \xi_2 e^{i\omega_2 t}$

$\theta(t)$ is an arbitrary phase term which could vary with time, in general.

$\omega_1 = 2\pi\nu_1$ and $\omega_2 = 2\pi\nu_2$ are the frequencies of emission from the two sources.

The E-fields received at points A and B in the aperture are (via Huygens-Fresnel superposition)

$$E_A = \mathcal{E}_1 e^{i\phi_{1A}} + \mathcal{E}_2 e^{i\phi_{2A}}, \quad E_B = \mathcal{E}_1 e^{i\phi_{1B}} + \mathcal{E}_2 e^{i\phi_{2B}} \tag{5}$$

Nithyanandan Thyagarajan (CSIRO)
26-Oct-2021

$e^{i\phi_{1A}}$, $e^{i\phi_{2A}}$, $e^{i\phi_{1B}}$, and $e^{i\phi_{2B}}$ are the propagators

$$\phi_{1A} = \frac{\omega_1}{c} R_{1A}\ , \quad R_{1A} = \text{distance}(P_1, Q_A), \ldots \tag{6}$$

Then,

$$\langle E_A E_B^* \rangle = \left\langle \left(\mathcal{E}_1 e^{i\phi_{1A}} + \mathcal{E}_2 e^{i\phi_{2A}} \right) \left(\mathcal{E}_1^* e^{-i\phi_{1B}} + \mathcal{E}_2^* e^{-i\phi_{2B}} \right) \right\rangle$$

$$\begin{aligned} &= \left\langle |\mathcal{E}_1|^2 e^{i(\phi_{1A} - \phi_{1B})} \right\rangle \\ &\quad + \left\langle |\mathcal{E}_2|^2 e^{i(\phi_{2A} - \phi_{2B})} \right\rangle \\ &\quad + \left\langle \mathcal{E}_1 \mathcal{E}_2^* e^{i(\phi_{1A} - \phi_{2B})} \right\rangle \\ &\quad + \left\langle \mathcal{E}_1^* \mathcal{E}_2 e^{i(\phi_{2A} - \phi_{1B})} \right\rangle \end{aligned} \tag{7}$$

$$\begin{aligned} \langle E_A E_B^* \rangle = \ & \underline{I_1 e^{i\frac{\omega_1}{c}\hat{s}_1 \cdot (\vec{r}_A - \vec{r}_B)}} + \underline{I_2 e^{i\frac{\omega_2}{c}\hat{s}_2 \cdot (\vec{r}_A - \vec{r}_B)}} \\ & + \langle \mathcal{E}_1 \mathcal{E}_2^* \rangle e^{i\left(\frac{\omega_1}{c}\hat{s}_1 \cdot \vec{r}_A - \frac{\omega_2}{c}\hat{s}_2 \cdot \vec{r}_B\right)} \\ & + \langle \mathcal{E}_1^* \mathcal{E}_2 \rangle e^{i\left(\frac{\omega_2}{c}\hat{s}_2 \cdot \vec{r}_A - \frac{\omega_1}{c}\hat{s}_1 \cdot \vec{r}_A\right)} \end{aligned} \tag{8}$$

In general, there are terms beyond that in the VCZ theorem. The last two terms denote the coherence of the fields on the emission plane. Only if they are incoherent, that is, $\langle \mathcal{E}_1 \mathcal{E}_2^* \rangle = \langle \mathcal{E}_1^* \mathcal{E}_2 \rangle = 0$ in the emission plane, the VCZ theorem is valid and reduces to the radio interferometry measurement equation.

Nithyanandan.Thyagarajan@csiro.au

Nithyanandan Thyagarajan (CSIRO)
26-Oct-2021

Let us inspect $\langle \mathcal{E}_1 \mathcal{E}_2^* \rangle$.

$$\langle \mathcal{E}_1 \mathcal{E}_2^* \rangle = \langle \xi_1 \xi_2^* e^{i[(\omega_1-\omega_2)t - \theta(t)]} \rangle$$

For this to vanish,

$$\omega_1 \neq \omega_2$$

or, $\theta(t) \neq \text{constant}$

Either of these imply that the emitted E-fields cannot be monochromatic. That is, the phase term $(\omega_1-\omega_2)t - \theta(t)$ must vary within the averaging interval. If $\theta(t)$ varies, then the emitted E-field can't be considered monochromatic.

If $\omega_1 = \omega_2$ and $\theta(t) = \text{constant}$, then the emitted E-fields, $\mathcal{E}_1$ and $\mathcal{E}_2$ are truly monochromatic, in which case,

$$\langle \mathcal{E}_1 \mathcal{E}_2^* \rangle = \langle \xi_1 \xi_2^* \rangle e^{-i\theta} \neq 0$$

$$\boxed{\text{monochromatic} \Updownarrow \text{coherence}}$$

and the E-fields are not incoherent!
That's why quasi-monochromatic is used.

3. Assumptions and implications of sky coherence

In obtaining the VCZ theorem, we simply enforce $\langle \mathcal{E}_i \mathcal{E}_j^* \rangle = 0$ if $i \neq j$ (incoherence) which eliminates the unfamiliar terms.

Nithyanandan Thyagarajan (CSIRO)
26-Dec-2021

Making the sky emission incoherent is what makes the visibility dependent only on $\vec{r}_A - \vec{r}_B$ (spatial stationarity). That is, the absolute positions do not matter but only the difference. This is what allows us to accumulate visibilities using Earth rotation synthesis. If stationarity cannot be assumed, the visibilities measured at different instances (even for the same $\vec{r}_A - \vec{r}_B$) will not be identical and cannot be accumulated.

How valid is the assumption of sky incoherence?

For most scenarios, it is valid because we do not expect the microscopic emission process to be identical or have knowledge of the field at other locations to be correlated. So, it is generally a good assumption to make.

So, emission plane $\longrightarrow$ incoherent

aperture plane $\longrightarrow$ fully / partially coherent
(point) / (extended)

When is it invalid?

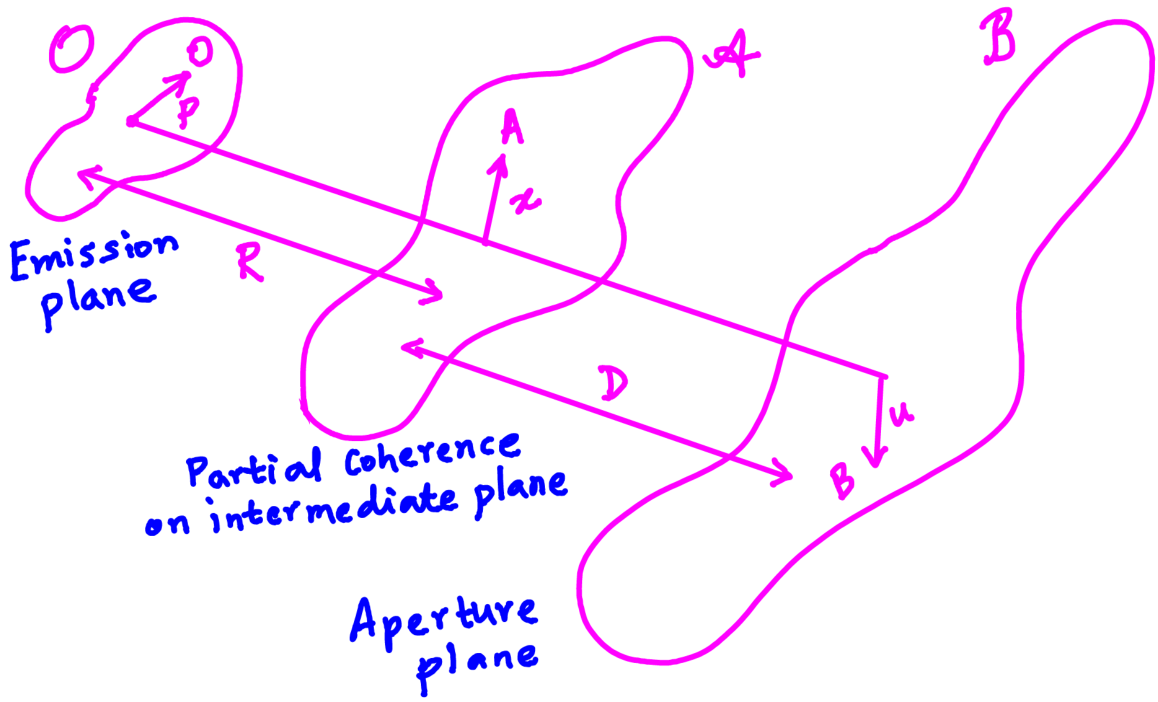
When there is an intermediate plane (e.g. lensing or scattering) that has acquired partial coherence and is now acting as the new emission plane which is received at the aperture plane.

Nithyanandan.Thyagarajan@csiro.au

Nithyanandan Thyagarajan (CSIRO)
26-Oct-2021

However, a treatment of mutual coherence and its propagation from one plane to another is possible [see "Principles of Optics" by Born & Wolf, "Statistical Optics" by Goodman, and the papers Van Cittert (1934) and Zernike (1938).]

Propagation of Mutual Coherence



The quantity, mutual coherence, is denoted by

$$J(P_1, P_2, \tau) = \langle E(P_1, t+\tau) E^*(P_2, t) \rangle \qquad (9)$$

Consider only mutual intensity (when $\tau = 0$).

The propagation of mutual intensity between any two planes P and Q is given by

$$J^{(Q)}(Q_1, Q_2) = \frac{1}{\lambda^2} \int_C \int_C J^{(P)}(P_1, P_2) \frac{e^{i\frac{2\pi}{\lambda}(s_1 - s_2)}}{s_1 s_2} \Lambda_1 \Lambda_2^* \, dP_1 \, dP_2 \qquad (10)$$

Nithyanandan.Thyagarajan@csiro.au

Nithyanandan Thyagarajan (CSIRO)
26-Oct-2021

Here, $s_1 \equiv P_1Q_1$, $s_2 \equiv P_2Q_2$. The VCZ theorem is a special case of this equation by setting $P_1 = P_2$ and imposing the incoherence criterion in plane P.

Consider three planes, $\mathcal{O}$, $\mathcal{A}$, and $\mathcal{B}$ as shown in the figure. The coherence functions in planes $\mathcal{O}$ and $\mathcal{B}$ can be related by

$$J^{(B)}(B_1, B_2) = \frac{1}{\lambda^4} \int_{\mathcal{O}} J^{(O)}(O_1, O_2)\, W(O_1, B_1)\, W^*(O_2, B_2)\, dO_1\, dO_2 \quad (11)$$

where,

$$W(O, B) = \int_A T(A) \frac{e^{i\frac{2\pi}{\lambda}(p+q)}}{pq}\, dA \quad (12)$$

is the dimensionless propagator from $\mathcal{O} \rightarrow \mathcal{B}$ and is defined on plane A using the transmission function, T(A), on plane A.

The factorizability of $J^{(B)}(B_1, B_2)$ into terms that depend on B_1 and B_2 is a test for the coherence of radiation incident on plane $\mathcal{A}$.

If $J^{(B)}(B_1, B_2)$ is sufficiently sampled, it can be used to infer W(O, B) and hence T(A) and this process is called holography of the scattering screen.

Nithyanandan.Thyagarajan@csiro.au

Nithyanandan Thyagarajan (CSIRO)
26-Dec-2021

4. Applications of mutual intensity & requirements

- What applications can we think of?
- What have we done so far?
- What else is required from modern arrays to explore these applications?
- Scattering screens (IPS, ISS)?
  (Temporal requirements severe?)
- Plasma/gravitational lensing?
  (Temporal requirements relaxed?)

At $\lambda \sim$ cm,

IPS : $t_s \sim 10-50$ ms, $\theta_s \sim 0\overset{''}{.}1 - 1''$

ISS : $t_s \sim 10-100$s, $\theta_s \sim 0\overset{''}{.}01 - 0\overset{''}{.}1$

- Require good aperture sampling on these timescales.
  $\Rightarrow$ VLBI is difficult for ISS but arrays like VLA may be used for IPS...
- Visibility accumulation timescale has to be significantly shorter than t_s ($\sim$ 1–10 ms)?
- Can SKA1-low, ASKAP, EDA2, MWA, etc. satisfy these requirements and be useful?
- Degeneracy between having a spatially coherent object and a scattering screen illuminated by an unresolved background.

Nithyanandan.Thyagarajan@csiro.au

- Be cautious about using self-calibration since the test for coherence involves a procedure similar to self-calibration of data in amplitude and phase by using a point source model.

Nithyanandan.Thyagarajan@csiro.au